Well known facts

A summary of results which you have either learnt or not but we do not prove here.

Theorem: Let μ be an outer measure on X. Then the measure generated by μ is a Borel measure iff

$\mu(A\cup B)=\mu(A)+\mu(B)$,

whenever $A,B\subset X$ and $d(A,B)>0$.

The proof is presented in the first Falconer book.

Definition: Let μ be a Borel measure on a separable metric space X. The support of μ $\operatorname{spt}\mu$ is th smallest closed set s.t. $\mu(X\setminus F)=0$.

$$\operatorname{spt}\mu = X\setminus\{x : \exists r>0,\ \text{s.t. } \mu(B(x,r))>0\}.$$

Example: $Q=\{q_1, q_2, \dots\}$ the set of rational numbers

$$\mu=\sum_{i=1}^{\infty} 2^{-i}\delta_{q_i} \qquad \delta_{q_i}(A)=\begin{cases}1 & \text{if } q_i\in A\\ 0 & \text{if } q_i\notin A\end{cases}$$

Then μ is a finite Radon measure on $\mathbb{R}$ with $\operatorname{spt}\mu=\mathbb{R}$ but μ is carried by the countable set Q in the sense that $\mu(\mathbb{R}\setminus Q)=0$.

Theorem (Well known) Let X be a separable metric space $f: X\to[0,\infty)$ Borel function. Then

$$\int f\,d\mu=\int_0^{\infty}\mu(\{x : f(x)\geq t\})\,dt.$$

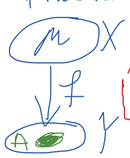
Riesz representation theorem

Let X be a locally cpt metric space & $L: C_0(X)\to\mathbb{R}$ a positive linear functional. Then $\exists!$ Radon measure μ s.t. $Lf=\int f\,d\mu$ for $f\in C_0(X)$,

where $C_0(X)$ is the space of compactly supported continuous real valued functions on X.

The push down measure

Sometimes people call it push forward measure.



$$\boxed{(f_*\mu)(A):=\mu(f^{-1}A)}$$

Change of variables formula

Theorem: Let $f: X\to Y$, be a Borel mapping, μ is a Borel measure on X $g: Y\to[0,\infty)$ Borel function. Then

$$\int_Y g\,d(f_*\mu)=\int_X g\circ f\,d\mu$$

Definition Weak convergence

Let $\mu_1, \mu_2, \dots, \mu, \dots$ be Radon measures on a metric space X. We say that $\mu_i \xrightarrow{w} \mu$ (that is μ_i converges weakly to μ) if

$$\lim_{n\to\infty}\int\varphi\,d\mu_i=\int\varphi\,d\mu$$

for all $\varphi\in C_0(X)$.

Examples:
- $\delta_i\to 0$ if $i\to\infty$
- $\mu_k := \frac{1}{k}\sum_{i=1}^{k}\delta_{i/k}$. Then $\mu_k \xrightarrow{w} \mathrm{Leb}|_{[0,1]}$.

The weak convergence is useful for:

Thm: $\mu_1, \mu_2, \dots$ Radon measures on $\mathbb{R}^d$ with $\sup\{\mu_i(K): i=1,2,\dots\}<\infty$ for all compact set $K\subset\mathbb{R}^n$. Then $\exists\{\mu_{i_k}\}$ weakly convergent subsequence.

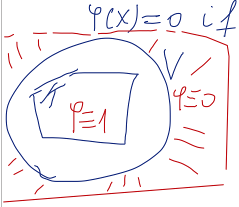
Theorem: Let $\mu, \mu_1, \mu_2, \dots$ be Radon measures on a locally cpt metric space. If $\mu_i \xrightarrow{w} \mu$ & $K\subset X$ cpt, $G\subset X$ open then

a) $\mu(K)\geq \limsup_{i\to\infty}\mu_i(K)$,

b) $\mu(G)\leq \liminf_{i\to\infty}\mu_i(G)$.

Proof of Part (a): $\forall\varepsilon>0$ $\exists$ $K\subset V$ open s.t. $\mu(V)<\mu(K)+\varepsilon$

By Urison's Lemma $\exists$ $\varphi\in C_0(X)$; $0\leq\varphi\leq 1$ s.t. $\varphi(x)=1$ if $x\in K$, $\varphi(x)=0$ if $x\notin V$



Thus

$$\mu(K)\geq\mu(V)-\varepsilon\geq\int\varphi\,d\mu-\varepsilon=\lim_{i\to\infty}\int\varphi\,d\mu_i-\varepsilon\geq\limsup_{i\to\infty}\mu_i(K)-\varepsilon. \blacksquare$$

The proof of part (b) is similar.

Now we prove the most important covering theorems:

Vitali CT for Leb. measure, Besicovitch CT, Vitali CT for Radon measures, Vitali CT for Hausdorff measures.

5r covering theorem

Let $B=B(x,r)$. We write $t\cdot B:=B(x,t\cdot r)$ in an Euclidean space.

Let $\mathcal{C}$ be a collection of balls contained in a bounded subset of $\mathbb{R}^n$. Then $\exists$ a finite or countable sub collection $\{B_i\}$ s.t.

$$\bigcup_{B\in\mathcal{C}} B\subset\bigcup_i 5\cdot B_i.$$

Proof: $d_0:=\sup\{\operatorname{diam}(B): B\in\mathcal{C}\}$

Let $B_1\in\mathcal{C}$ s.t. $\operatorname{diam}(B)\geq\frac{d_0}{2}$.

If $B_1,\dots,B_m$ have already been chosen let

$$d_m:=\sup\{|B| : B\in\mathcal{C},\ B\cap\bigcup_1^m B_i=\emptyset\}$$

$|B|=\operatorname{diam}(B)$

If $d_m=0$ then we are ready otherwise we choose $B_{m+1}\in\mathcal{C}$ s.t.

- $B_{m+1}\cap\bigcup_1^m B_i=\emptyset$,
- $|B_{m+1}|\geq\frac{1}{2}d_m$.

We claim that

$$\bigcup_{B\in\mathcal{C}} B\subset\bigcup_i 5B_i \quad (*)$$

Namely, if $B\in\mathcal{C}$ and $B\neq B_i$ $\forall i$ then $\exists i$ s.t. (***)

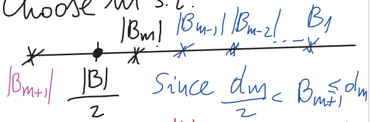
$$B\cap B_i\neq\emptyset \text{ and } 2|B_i|\geq|B|$$

First we verify (***).

Recall: for every $m\geq1$:

$$d_m=\sup\{|B'| : B'\in\mathcal{B},\ B'\cap\bigcup_1^m B_i=\emptyset\}$$

Choose m s.t.



Since $\frac{d_m}{2}<B_{m+1}\leq d_m$

$$\frac{d_m}{2}<|B_{m+1}|\leq\frac{|B|}{2}\Longrightarrow d_m<|B|.$$

So, $\exists$ $i\in\{1,\dots,m\}$ $B\cap B_i\neq\emptyset$ but for all such i: $|B_i|\geq\frac{|B|}{2}$.

So (***) has been verified.

Note that $|B_i|\to 0$ since $\sum|B_i|^d<\infty$. However, (***) implies that $B\subset 5B_i$ which completes the proof.

Vitali Covering Theorem

Let $A\subset\mathbb{R}^d$ and $\mathcal{B}$ is a family of closed balls in $\mathbb{R}^d$ s.t. every $x\in A$ is contained in some arbitrarily small elements of $\mathcal{B}$. That is

$$\inf\{\operatorname{diam}(B) \mid x\in B\in\mathcal{B}\}=0.$$

Then $\exists \{B_i\}$, $B_i \in \mathcal{B}$ s.t.

- $B_i \cap B_j = \emptyset$ if $i \neq j$,
- $\mathcal{L}eb_n(A \setminus \bigcup_i B_i) = 0$.
- $\forall \varepsilon > 0$ we can choose $\{B_i\}$ s.t. $\sum_i \mathcal{L}eb_n(B_i) \le \mathcal{L}eb_n(A) + \varepsilon$.

Proof: Assume that A is bounded. Choose $A \subset U$ open set s.t.
$\mathcal{L}eb_d(U) \le (1+7^{-d})\,\mathcal{L}eb_d(A)$.
Let $\mathcal{B}$ be the collection of all balls contained in U. Let $\{B_i\}$ be a finite or countable subcollection of $\mathcal{B}$ chosen according to the $5r$ covering theorem:
$A \subset \bigcup_i 5B_i$, $B_i \subset U$

So
$$5^{-d}\mathcal{L}eb_d(A) \le 5^{-d}\sum_i \mathcal{L}eb_d(5B_i) = \sum_i \mathcal{L}eb_d(B_i)$$
Hence we can find k_1 s.t.
$$6^{-d}\mathcal{L}eb_d(A) \le \sum_{i=1}^{k_1}\mathcal{L}eb_d(B_i)$$
$A_1 := A \setminus \bigcup_{i=1}^{k_1} B_i$

let $u := 1+7^{-d} - 6^{-d} < 1$

$\mathcal{L}eb_d(A_1) \le u \cdot \mathcal{L}eb_d(A)$.

Namely,
$$\mathcal{L}eb_d(A_1) \le \mathcal{L}eb_d\Big(U \setminus \bigcup_{i=1}^{k_1} B_i\Big) = \mathcal{L}eb_d(U) - \sum_{i=1}^{k_1}\mathcal{L}eb_d(B_i) \le (1+7^{-d}-6^{-d})\,\mathcal{L}eb_d(A) = u\,\mathcal{L}eb_d(A).$$
Remember: B_i are closed balls!
Since $A_1 \subset \left(\bigcup_1^{k_1} B_i\right)^c$, $\exists U_1$ open
$A_1 \subset U_1$, $\mathcal{L}eb_d(U_1) \le (1+7^{-d})\,\mathcal{L}eb_d(A_1)$
$\boxed{U_1 \cap \bigcup_{i=1}^{k_1} B_i = \emptyset}$

As above: $\exists \{B_i\}_{i=k_1+1}^{k_2}$ disjoint balls $B_i \in \mathcal{B}$, $B_i \subset U_1$ s.t. for
$$A_2 = A_1 \setminus \bigcup_{i=k_1+1}^{k_2} B_i = A \setminus \bigcup_{i=1}^{k_2} B_i$$
$\mathcal{L}eb_d(A_2) \le u\,\mathcal{L}eb_d(A_1) \le u^2\,\mathcal{L}eb_d(A)$.
Clearly all of the balls $\{B_i\}_{i=1}^{k_2}$ are disjoint.

After m step we get $\{B_i\}_{i=1}^{k_m}$, $B_i \in \mathcal{B}$ disjoint balls.
$$\mathcal{L}eb_d\Big(A \setminus \bigcup_1^{k_m} B_i\Big) \le u^m \cdot \mathcal{L}eb_d(A).$$
Using that $u<1$ the result follows. If A is NOT bounded: Partition $\mathbb{R}^d$ into unit cubes with disjoint interior. Apply the result for the part of A in such a cube and observe that the intersection of the cubes have zero measure.

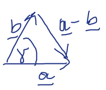
Remark 1.) Assume that μ is a Radon measure for which $\exists\, 1<\tau<\infty$:
$$\lim_{r\to 0}\left\{\frac{\mu(B(y,\tau r))}{\mu(B(y,r))} : x \in B(y,r)\right\} < \infty,$$
for μ-a.e. $x \in \mathbb{R}^d$!
Then we can substitute the Leb measure for μ. The proof is the same.

Remark 2.) There are Radon measures for which the theorem does not hold:
$A \subset \mathbb{R}^2$, $\mu(A) := \mathcal{L}eb_1\{x \in \mathbb{R} : (x,0) \in A\}$.
$\mathcal{B} = \{B((x,y),y) : x \in \mathbb{R}, 0<y<\infty\}$
Covers $A = \{(x,0) \mid x \in \mathbb{R}\}$ satisfying the requirements of the Thm but there is no way to find a countable subfamily which covers μ-a.e. points of A.

Besicovitch Covering theorem

First we need two lemmas:

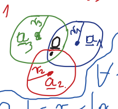
Lemma: Let $\underline{a}, \underline{b} \in \mathbb{R}^2$ s.t.
(i) $0 < |\underline{a}| < |\underline{a}-\underline{b}|$
(ii) $0 < |\underline{b}| < |\underline{a}-\underline{b}|$



Then the angle between $\underline{a}, \underline{b}$ is at least $60°$. That is
$$\left|\frac{\underline{a}}{|\underline{a}|} - \frac{\underline{b}}{|\underline{b}|}\right| \ge 1.$$
Proof: The proof is elementary and everybody proves it alone.

Lemma: $\exists$ a constant $N = N(d)$ depending only on d s.t.

Suppose that $\exists k$, $\underline{a}_1,\dots,\underline{a}_k \in \mathbb{R}^d$ & $r_1,\dots,r_k \in \mathbb{R}^+ = (0,\infty)$ s.t.
$\forall i\neq j$, $\underline{a}_i \notin B(\underline{a}_j, r_j)$, $\bigcap_{i=1}^k B(\underline{a}_i, r_i) \neq \emptyset$
Then $k \le N(d)$.



(*) $\forall i \neq j$ $|\underline{a}_i| < r_i < |\underline{a}_i - \underline{a}_j|$

Proof: Assume that
(a) $\underline{a}_i \neq \underline{0}$ $\forall i$, and
(b) $\underline{0} \in \bigcap_{i=1}^k B(\underline{a}_i, r_i)$.

Consider the plane spanned by $\underline{a}_i, \underline{a}_j$, where $i \neq j$ fixed, and apply the previous lemma with $\boxed{\underline{a} = \underline{a}_i \ \&\ \underline{b} = \underline{a}_j}$ we get:
$$\left|\frac{\underline{a}_i}{|\underline{a}_i|} - \frac{\underline{a}_j}{|\underline{a}_j|}\right| \ge 1, \quad i \neq j \qquad (**)$$

So the points $\frac{\underline{a}_i}{|\underline{a}_i|} \in S^{d-1}$ (unit sphere in $\mathbb{R}^d$) s.t. the distance between any two of them is at least 1. S^{d-1} is compact so, $\exists$ an upper bound $N(d)$ for the number of this points. □

Besicovitch's covering theorem
$\exists P(d), Q(d)$ constants with the following properties:
Given • $A \subset \mathbb{R}^d$ bounded,
• $\mathcal{B}$ family of closed balls s.t. $\forall a \in A$, $\exists B \in \mathcal{B}$ centered at a.
Then

(a) $\exists$ a finite or countable collection of balls $\{B_i\}$, $B_i \in \mathcal{B}$ s.t.
(*) $\boxed{\mathbb{1}_A \le \sum_i \mathbb{1}_{B_i} \le P(d)}$
(b) $\exists$ subfamilies $\mathcal{B}_1,\dots,\mathcal{B}_{Q(d)} \subset \mathcal{B}$ covering A such that $\mathcal{B}_i$ consists of disjoint balls.
(**) $\boxed{A \subset \bigcup_{i=1}^{Q(d)} \bigcup_{B \in \mathcal{B}_i} B}$,
(***) $B \cap B' = \emptyset$, $\forall B, B' \in \mathcal{B}_i$, if $B \neq B'$

Proof: $\forall x \in A$ pick an $r(x)$ s.t. $B(x, r(x)) \in \mathcal{B}$. Since A is bounded we may assume that
$M_1 = \sup r(x) < \infty$. (Otherwise we have $B \in \mathcal{B}$ which covers A alone)
Inductively, we choose $x \in A$ with $r(x_1) > \frac{M_1}{2}$. If we have already chosen $x_1,\dots,x_j$ let $x_{j+1} \in A \setminus \bigcup_{i=1}^{j} B(x_i, r(x_i))$, with $r(x_{j+1}) \ge \frac{M_1}{2}$ as long as possible. Since A is bounded the process terminates in finitely many steps. In this way we obtained: $x_1, x_2, \dots, x_{k_1}$.

Let $M_2 := \sup\{r(x) : x \in A \setminus \bigcup_{i=1}^{k_1} B(x_i, r(x_i))\}$
Choose $x_{k_1+1} \in A \setminus \bigcup_{i=1}^{k_1} B(x_i, r(x_i))$ with $r(x_{k_1+1}) \ge M_2/2$. Further, $x_{j+1} \in A \setminus \bigcup_{i=1}^{j} B(x_i, r(x_i))$; with $r(x_{j+1}) \ge M_2/2$.

Inductively, this results $0 = k_0 < k_1 < k_2 < \dots$ & $M_i \downarrow 0$ with $2M_{i+1} < M_i$ and we get a sequence of balls
$B_i = B(x_i, r(x_i)) \in \mathcal{B}$,
$I_j = \{k_{j-1}+1, \dots, k_j\}$, $j = 1,2,\dots$
(i) $M_j/2 \le r(x_i) \le M_j$ if $i \in I_j$.
(ii) $x_{j+1} \in A \setminus \bigcup_{i=1}^{j} B_i$, $j = 1,2,\dots$
(iii) $x_i \in A \setminus \bigcup_{m \neq k} \bigcup_{j \in I_m} B_j$, $i \in I_k$.
It is obvious that (i) & (ii) holds. To see (iii):

$j \in I_m$, $i \in I_k$.
- if $k<m$, $x_j \notin B_i$ by (ii)
- if $m<k$, then $r(x_j) < r(x_i)$ & $x_i \notin B_j$ by (ii). Hence $x_i \notin B_j$.

(iii) means that balls from different blocks do NOT contain each others centers.
We know that $M_i \downarrow 0$ ($2M_{i+1} \le M_i$)
So, by (i): $r(x_i) \to 0$. By the construction: $A \subset \bigcup_{i=1}^{\infty} B_i$.
To establish the second inequality in (*), assume that $x \in \bigcap_{i=1}^{p} B_{m_i}$. We prove that
(***) $p \le P(d) = 16^d \cdot N(d)$ (the constant defined in the previous lemma)

Namely, by (iii) and the previous Lemma the indices m_i can belong at most $N(d)$ different blocks I_j. So, we need to prove that

$$\#\{I_j \cap \{m_i : i=1,\dots,p\}\} \le 16^d$$

To see this fix j and write $\{l_1,\dots l_q\} := I_j \cap \{m_i : i=1,\dots,p\}$.

By (i) & (ii) the balls:

$$B\left(x_{l_i}, \tfrac{1}{4} r(x_{l_i})\right), \quad i=1,\dots,q$$

are disjoint and

$$\bigcup_{i=1}^{q} B\left(x_{l_i}, \tfrac{1}{4} r(x_{l_i})\right) \subset B(x, 2M_j).$$

Hence, for $\alpha_d := \mathrm{Leb}_d(B(0,1))$

$$q \cdot \alpha_d \cdot \left(\frac{M_j}{8}\right)^d \le \sum_{i=1}^{q} \mathrm{Leb}_d\left(B(x_{l_i}, r(x_{l_i}))\right) \le \mathrm{Leb}_d(B(x, 2M_j)) = \alpha_d \cdot (2M_j)^d$$

This implies that $q \le 16^d$ which completes the proof of part (a).

Now we prove part (b): Let $\{B_i\}_{i=1}^{\infty}$ be the balls constructed above and we write $B_i = B(x_i, r_i)$. We know that $r_i \to 0$. We may assume that $r_1 \ge r_2 \ge \dots$ Let $B_{1,1} := B_1$. Assume that $B_{1,1}, \dots, B_{1,j}$ have already been chosen.

Then $B_{1,j+1} := B_k$ if k is the smallest integer with

$$B_k \cap \bigcup_{i=1}^{j} B_{1,i} = \emptyset. \quad \text{Let}$$

$$\mathcal{B}_1 := \{B_{1,1}, B_{1,2}, B_{1,3}, \dots\}$$

If A is not covered by $\mathcal{B}_1$ then we define $B_{2,1}$ as the smallest B_k for which $B_k \notin \mathcal{B}_1$. Inductively, $B_{2,j+1}$ is the smallest B_k s.t. $B_k \cap \bigcup_{i=1}^{j} B_{2,i} = \emptyset$. In this way we define $\mathcal{B}_1, \mathcal{B}_2, \dots$ such that each of them are disjoint.

We claim that:

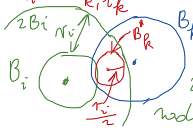
$$(*) \quad A \subset \bigcup_{k=1}^{m} \bigcup_{B \in \mathcal{B}_k} B, \quad m \le 4^d \cdot P(d) + 1.$$

To see this we show that

$$(**) \quad x \in A \setminus \bigcup_{k=1}^{m} \bigcup_{B \in \mathcal{B}_k} B \Longrightarrow m \le 4^d \cdot P(d)$$

$\exists i$ s.t. $x \in B_i$. Then $\forall k = 1,\dots,m$ $B_i \notin \mathcal{B}_k$. By the construction of $\mathcal{B}_k$, $\exists i_k$:

$B_i \cap B_{k,i_k} \ne \emptyset$ & $r_i \le r_{k,i_k}$



B_{k,i_k}. So $\exists B'_k$ with center in $2B_i \cap B_{k,i_k}$ and radius $r_i/2$.

We know that each point in $\mathbb{R}^d$ is contained in at most $P(d)$ balls of the form B_{k,i_k} so the same is true for the balls B'_k. That is

$$\sum_{k=1}^{m} \mathbb{1}_{B'_k} \le P(d) \mathbb{1}_{\bigcup_{k=1}^{m} B'_k}$$

Using that $B'_k \subset 2B_i$:

$$2^d \alpha_d r_i^d = \mathrm{Leb}_d(2B_i) \ge \mathrm{Leb}_d\left(\bigcup_{k=1}^{m} B'_k\right)$$

$$= \int \mathbb{1}_{\bigcup_{k=1}^{m} B'_k}(x)\, dx \ge P(d)^{-1} \int \sum_{k=1}^{m} \mathbb{1}_{B'_k}(x)\, dx$$

$$= P(d)^{-1} \sum_{k=1}^{m} \mathrm{Leb}_d(B'_k)$$

$$\ge m\, P(d)^{-1} 2^{-d} \alpha_d \cdot r_i^d.$$

Hence $m \le 4^d P(d)$. $\square$

This completes the proof of the Besicovitch's theorem.

Now we prove Vitaly's covering thm. for Radon measures.

Theorem (Vitali's covering theorem for Radon measure)

Let μ be a Radon measure on $\mathbb{R}^d$, $A \subset \mathbb{R}^d$ and $\mathcal{B}$ is a family of closed balls s.t.

$$\forall x \in A: \inf\{r : B(x,r) \in \mathcal{B}\} = 0.$$

(every point of A is contained in an arbitrary small element of $\mathcal{B}$ centered at x.)

Then $\exists \{B_i\}$, $B_i \in \mathcal{B}$ disjoint balls s.t. $\mu(A \setminus \bigcup_i B_i) = 0$.

Proof: We may assume that $\mu(A) > 0$. First we suppose that A is bounded.

Using that μ is a regular measure $\exists\, U$ open set with $\mu(U) \le \left(1 + \frac{1}{4Q(d)}\right)\mu(A)$, and $A \subset U$

This is the constant from Besicovitch's Theorem

Using the notation of Besicovitch's thm:

$$A \subset \bigcup_{i=1}^{Q(d)} \bigcup_{B \in \mathcal{B}_i} B \subset U. \quad \text{Hence}$$

$$\mu(A) \le \sum_{i=1}^{Q(d)} \mu\left(\bigcup_{B \in \mathcal{B}_i} B\right). \text{ So } \exists i:$$

$$\mu(A) \le Q(d) \cdot \mu\left(\bigcup_{B \in \mathcal{B}_i} B\right)$$

We can find a finite subfamily $\mathcal{B}'_i \subset \mathcal{B}_i$ such that

$$\mu(A) \le 2\, Q(d)\, \mu\left(\bigcup_{B \in \mathcal{B}'_i} B\right)$$

Let $A_1 := A \setminus \bigcup_{B \in \mathcal{B}'_i} B$. Then

$$\mu(A_1) \le \mu\left(U \setminus \bigcup_{B \in \mathcal{B}'_i} B\right) = \mu(U) - \mu\left(\bigcup_{B \in \mathcal{B}'_i} B\right)$$

$$\le \left(1 + \tfrac{1}{4} Q(d)^{-1} - \tfrac{1}{2} Q(d)^{-1}\right) \cdot \mu(A)$$

$$= u \cdot \mu(A), \quad \text{for } u = 1 - \frac{1}{4Q(d)} < 1.$$

The proof is finished exactly as the proof of Vitali's covering thm for Leb. measure.

If A is not bounded we divide $\mathbb{R}^d$ into cubes so that their faces are of zero μ-measure sets.

Now we turn to Hausdorff measures.

Theorem

(a) For every $E \subset \mathbb{R}^d$, $\exists\, G \supset E$ which is a G_δ-set satisfying $E \subset G$ and $\mathcal{H}^s(G) = \mathcal{H}^s(E)$.

(b) If E is an $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(E) < \infty$ then $\forall \varepsilon > 0$, $\exists F$ F_σ set, $F \subset E$ and $\mu(E \setminus F) < \varepsilon$.

Proof (a) If $\mathcal{H}^s(E) = \infty$ then $G = \mathbb{R}^d$.

Suppose that $\mathcal{H}^s(F) < \infty$.

For each $i = 1, 2, 3, \dots$ we choose a $\frac{2}{i}$ cover $\{U_{ij}\}_{j=1}^{\infty}$

$$E \subset \bigcup_{j=1}^{\infty} U_{ij}; \quad |U_{ij}| < \delta \text{ s.t.}$$

$$\sum_{j=1}^{\infty} |U_{ij}| < \mathcal{H}^s_{\frac{1}{i}}(E) + \frac{1}{i}$$

Let $G := \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} U_{ij}$. Then G is G_δ set and $E \subset G$.

Further, $\mathcal{H}^s_{\frac{2}{i}}(G) \le \mathcal{H}^s_{\frac{1}{i}}(E) + \frac{1}{i}$.

Hence $\mathcal{H}^s(E) = \mathcal{H}^s(G)$.

Using that G_δ sets are Borel sets and Borel sets are $\mathcal{H}^s|_E$-measurable. $\mathcal{H}^s$ is a regular outer measure.

(b) Assume that $E \subset \mathbb{R}^d$ is an $\mathcal{H}^s$-measurable set with $\mathcal{H}^s(E) < \infty$. Using (a) we can find open sets $O_1, O_2, \dots$, $O_i \supset E$ & $0 = \mathcal{H}^s\left(\bigcap_{i=1}^{\infty} O_i \setminus E\right) = \mathcal{H}^s(\cap O_i) - \mathcal{H}^s(E)$.

$\exists F_{i,j} \nearrow O_i$, $F_{i,j}$ closed $\{F_{i,j}\}_j$ is increasing $\bigcup_{j=1}^{\infty} F_{i,j} = O_i$. So,

$$\lim_{j \to \infty} \mathcal{H}^s(E \cap F_{i,j}) = \mathcal{H}^s(E \cap O_i) = \mathcal{H}^s(E).$$

That is for any $\varepsilon > 0$ $\exists \{j_i\}_i$:

$$\mathcal{H}^s(E \setminus F_{i,j_i}) < 2^{-i} \varepsilon \quad (i = 1, 2, \dots)$$

Let $F := \bigcap_{i=1}^{\infty} F_{i,j_i}$. Then

$$\mathcal{H}^s(F) \ge \mathcal{H}^s(E \cap F) \ge$$
$$\ge \mathcal{H}^s(E) - \sum_{i=1}^{\infty} \mathcal{H}^s(E \setminus F_{i,j_i})$$
$$\ge \mathcal{H}^s(E) - \varepsilon.$$

We know that $F \subset \bigcap_{i=1}^{\infty} O_i$ hence

$$\mathcal{H}^s(F \setminus E) \le \mathcal{H}^s\left(\bigcap_{i=1}^{\infty} O_i \setminus E\right) = 0.$$

By (a) $F \setminus E$ is contained in some G_δ-set G with $\mathcal{H}^s(G) = 0$. Hence $F \setminus G \subset E$ & $F \setminus G$ is an F_σ-set

$$\mathcal{H}^s(F \setminus G) \ge \mathcal{H}^s(F) - \mathcal{H}^s(G) > \mathcal{H}^s(E) - \varepsilon.$$

For each $\varepsilon = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ we construct such an F_σ-set and consider their countable union which is contained in E and of equal meas. to E.

Lemma: Let $E \subset \mathbb{R}^d$ be $\mathcal{H}^s$-measurable with $\mathcal{H}^s(E) < \infty$ and let $\varepsilon > 0$. Then $\exists$ $\rho = \rho(E, \varepsilon)$ s.t. for any collection of Borel sets $\{U_i\}_{i=1}^{\infty}$ with $0 < |U_i| < \sum_i |U_i|^s + \varepsilon$ $(*)$

Proof: From the definition of $\mathcal{H}^s$, $\exists \rho > 0$ s.t.

$$\mathcal{H}^s(E) < \sum_i |W_i|^s + \frac{1}{2}\varepsilon$$

for any ρ-cover $\{W_i\}$ of E.

Given Borel sets $\{U_i\}$ with $0 < |U_i| \le \rho$. Then $\exists$ ρ-cover $\{V_i\}$ of $E \setminus \bigcup_i U_i$ s.t.

$$\mathcal{H}^s\left(E \setminus \bigcup_i U_i\right) + \frac{1}{2}\varepsilon > \sum_i |V_i|^s$$

Note that $\{U_i\} \cup \{V_i\}$ is a ρ-cover of E.

$$\mathcal{H}^s(E) < \sum_i |U_i|^s + \sum_i |V_i|^s + \frac{1}{2}\varepsilon.$$

Put together the last two formulas to get: if $\sum_i |U_i|^s = \infty$ then $(*)$ is obvious.

Otherwise:

$$\mathcal{H}^s\left(E \cap \bigcup_i U_i\right) = \mathcal{H}^s(E) - \mathcal{H}^s\left(E \setminus \bigcup_i U_i\right)$$
$$\le \sum_i |U_i|^s + \sum_i |V_i|^s + \frac{1}{2}\varepsilon - \sum_i |V_i|^s + \frac{1}{2}\varepsilon = \sum_i |U_i|^s + \varepsilon.$$

Definition: (Vitali class) A collection of closed sets $\mathcal{V}$ is called Vitali class for the set $E \subset \mathbb{R}^d$ if $\forall x \in E$, $\forall \delta > 0$, $\exists V \in \mathcal{V}$ with $x \in V$ and $0 < |V| \le \delta$.

Tétel: Vitali CT for Hausdorff measure

(a) Let $E \subset \mathbb{R}^d$ be $\mathcal{H}^s$-measurable and $\mathcal{V}$ be a Vitali class of closed sets for E. Then we may select a finite or countable disjoint sequence $\{U_i\}$ from $\mathcal{V}$ s.t. either $(*)$

$$\sum_i |U_i|^s = \infty \text{ or } \mathcal{H}^s\left(E \setminus \bigcup_i U_i\right) = 0$$

(b) If $\mathcal{H}^s(E) < \infty$ then for any $\varepsilon > 0$ we may require that $\mathcal{H}^s(E) \le \sum_i |U_i|^s + \varepsilon$.

<u>Proof:</u> Fix $\rho > 0$. We may assume that $|U| \leq \rho$ for all $U \in V^s$.

Let $U_1 \in V^s$ be arbitrary. Assume that $U_1, \dots, U_m$ have already been chosen and

$$d_m := \sup\{|U| : U \in V^s, \ U \cap \bigcup_{i=1}^{m} U_i = \emptyset\}$$

If $d_m = 0$, then $E \subset \bigcup_1^m U_i$ then (a) follows and the process terminates. If $d_m > 0$, $U_{m+1} \in V^s$ is chosen s.t. $U_{m+1} \cap \bigcup_1^m U_i = \emptyset$ and $|U_{m+1}| > \frac{1}{2} d_m$. Suppose that a process continues indefinitely s.t. $\sum_i |U_i|^s < \infty$. If this does not hold we are ready.

For every i, let $x_i \in U_i$ be arbitrary and we define $B_i := B(x_i, 3|U_i|)$.

<u>Claim:</u> $E \setminus \bigcup_1^k U_i \subset \bigcup_{i=k+1}^{\infty} B_i$.

<u>Proof of the Claim:</u> Let $x \in E \setminus \bigcup_{i=1}^{k} U_i$. We can choose $U \in V^s$ s.t.

$$x \in U \ \& \ U \cap \bigcup_{i=1}^{k} U_i = \emptyset.$$

U_i are closed

Since $\sum |U_i|^s < \infty$ we have $|U_i| \to 0$. So $\exists m$ s.t. $|U| > 2|U_m|$. Then $\exists\ k < i < m$ for which

$$U \cap U_i \neq \emptyset \ \& \ |U| \leq 2|U_i|.$$

Clearly $U \subset B_i$ so the Claim is ✓. Proved!

Thus if $\delta > 0$

$$\mathcal{H}^s_\delta\Big(E \setminus \bigcup_{i=1}^{\infty} U_i\Big) \leq \mathcal{H}^s_\delta\Big(E \setminus \bigcup_{i=1}^{k} U_i\Big) \leq \sum_{i=k+1}^{\infty} |B_i|^s = 6^s \cdot \sum_{i=k+1}^{\infty} |U_i|^s,$$

whenever k is so large that $\forall i > k$: $|B_i| \leq \delta$ holds. Hence

$\mathcal{H}^s_\delta(E \setminus \bigcup_{i=1}^{\infty} U_i) = 0$, $\forall \delta > 0$. So,

$\mathcal{H}^s(E \setminus \bigcup_{i=1}^{\infty} U_i) = 0$ which proves (a).

To prove (b) we may assume that the ρ chosen at the beginning of the proof is the $\rho = \rho(\varepsilon, E)$, from the previous Lemma.

If $\sum_{i=1}^{\infty} |U_i|^s = \infty$ the (b) follows.

Assume that $\sum_{i=1}^{\infty} |U_i|^s < \infty$. Then by the previous Lemma:

$$\mathcal{H}^s(E) = \mathcal{H}^s\Big(E \setminus \bigcup_i U_i\Big) + \mathcal{H}^s\Big(E \cap \bigcup_i U_i\Big)$$

$$= 0 + \mathcal{H}^s\Big(E \cap \bigcup_i U_i\Big)$$

$$< \sum_i |U_i|^s + \varepsilon. \quad \square$$